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The area of intersection of n equal circular disks

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Abstract. An analytical expression is given for the area of intersection of three equal (circular) disks as a function of the distances. Then it is shown that the intersection of n disks can always be reduced to intersections of less than four disks in a definite manner. Some physical applications are also discussed.

1. Introduction

For the evaluation of the equation of state of a hard-disk fluid it is advantageous to calculate as many virial coefficients as possible (Rowlinson 1964, Hemmer 1965, Kratky 1976). The virial coefficients consist of cluster integrals the integrands of which can be interpreted as (products of) intersections of disks. These formal disks have radius d if the original hard disks had diameter d. Thus, a better knowledge of the intersection of n disks also has physical consequences.

In the following, n open disks with unit radius will be considered. The designation of the disks corresponds to the designation of their centres. For instance, centre 1 (or 'point' 1) is the centre of disk 1. The geometrical areas of the disks lie within the same plane so that the concept of intersection makes sense.

Definition 1.1. The intersection of n disks has the following three possible meanings in the present paper.

(i) The set $\mathrm{I}^{\mathrm{s}}(1, 2, \ldots, n) \doteqdot \{p/r_{pj} < 1, j = 1, 2, \ldots, n\}$, the points j being fixed at definite, but arbitrary positions.

(ii) The measure of I^{s}, $\mathrm{I}(1, 2, \ldots, n)$, the area in the sense of integration theory. $\mathrm{I}(1, 2, \ldots, n)$ is a continuous function of the location of the centres $1, 2, \ldots, n$, with symmetrical indices, and it is only a function of the relative distances. Therefore, one can write for example $\mathrm{I}(r_{12}, r_{13}, r_{23})$ instead of $\mathrm{I}(1, 2, 3)$.

(iii) The geometrical area which is characterised by a certain structure of the boundary. This area, which is always convex, will be denoted $\mathrm{I}^{\mathrm{g}}(1, 2, \ldots, n)$.

The order of the numbers of the disks is in principle irrelevant, e.g. $\mathrm{I}(1, 2, 3) = \mathrm{I}(1, 3, 2)$. For convenience, a monotonically increasing sequence will be chosen in most cases.

From the definition of $\mathrm{I}^{\mathrm{s}}(1, 2, \ldots, n)$ it follows that

$$\mathrm{I}^{\mathrm{s}}(1, 2, \ldots, n) = \bigcap_{j=1}^{n} \mathrm{I}^{\mathrm{s}}(j). \tag{1.1}$$

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A typical application of this is:

$$[\mathrm{I}^s(1,2,3)=\mathrm{I}^s(2,3)]\Rightarrow[\mathrm{I}^s(1,2,\ldots,n)=\mathrm{I}^s(2,3,\ldots,n)]. \tag{1.2}$$

The same relation holds for I since

$$[\mathrm{I}^s(a)\subseteq\mathrm{I}^s(b)]\Rightarrow[\mathrm{I}(a)\leqslant\mathrm{I}(b)], \qquad [\mathrm{I}^s(a)=\mathrm{I}^s(b)]\Rightarrow[\mathrm{I}(a)=\mathrm{I}(b)], \tag{1.3}$$

where $\{a\}$ and $\{b\}$ are sets of numbers representing the centres of disks at definite locations. From (1.1) it follows that

$$[\{a\}\supseteq\{b\}]\Rightarrow[\mathrm{I}^s(a)\subseteq\mathrm{I}^s(b)]. \tag{1.4}$$

Definition 1.2. The following notation for I is used in this paper (the geometrical area I^g will be designated in the same way).

$$[\mathrm{I}^s(1,2,\ldots,n)=\mathrm{I}^s(i)]\Leftrightarrow\mathrm{I}(1,2,\ldots,n)=\mathrm{I}_i$$

$$\begin{bmatrix}\mathrm{I}^s(1,2,\ldots,n)=\mathrm{I}^s(i,j)\neq\varnothing\\ \mathrm{I}^s(i,j)\subset\mathrm{I}^s(i),\,\mathrm{I}^s(i,j)\subset\mathrm{I}^s(j)\end{bmatrix},\Leftrightarrow\mathrm{I}(1,2,\ldots,n)=\mathrm{I}_{ij}, \qquad 1\leqslant(i\neq j)\leqslant n$$

$$\vdots$$

$$[\mathrm{I}^s(1,2,\ldots,n)\neq\varnothing \text{ is not reducible as above}]\Leftrightarrow\mathrm{I}(1,2,\ldots,n)=\mathrm{I}_{12\ldots n}$$

$$[\mathrm{I}^s(1,2,\ldots,n)=\varnothing]\Leftrightarrow\mathrm{I}(1,2,\ldots,n)=0.$$

The fact that the disks have been defined as open guarantees that from $\mathrm{I}=0$ follows $\mathrm{I}^s=\varnothing$. The above notation has the advantage that one can deduce the essential properties of I^s and I^g from the designation of I. For instance, $\mathrm{I}_{12\ldots n}$ is an intersection where all the disks really contribute. It corresponds to a geometrical area which is bounded by n arcs, each coming from one of the disks.

In this paper, the lemmas will be stated without proof. The proofs of the theorems will be outlined.

2. The intersection of two and of three disks

Two disks of unit radius have the following intersection:

$$\mathrm{I}(1,2)=\begin{cases}\mathrm{I}_1=\mathrm{I}_2 & r_{12}=0\\ \mathrm{I}_{12} & 0<r_{12}<2\\ 0 & 2\leqslant r_{12},\end{cases} \tag{2.1}$$

where $\mathrm{I}_1=\pi$, $\mathrm{I}_{12}=2\cos^{-1}(r_{12}/2)-\frac{1}{2}r_{12}(4-r_{12}^2)^{1/2}$. This well known result is stated for instance by Lee *et al* (1969). $\mathrm{I}(1,2)$ is a continuous, monotonically decreasing function of r_{12}.

In the following, we turn to the intersection of three disks, which has been calculated by Hemmer (1965) in the case $r_{12}=1$. In the present paper, the general case is considered. If $r_{12}\geqslant 2$, combination of (1.4) and (2.1) yields $\mathrm{I}(1,2,3)=0$. If $r_{12}=0$, $\mathrm{I}(1,2,3)=\mathrm{I}(1,3)$, which has already been studied. Thus, only the case $0<r_{12}<2$ will be considered further. One can divide the possible location of centre 3 into several regions, where $\mathrm{I}(1,2,3)$ equals I_{12}, I_{13}, I_{23}, I_{123}, or zero, respectively. Ree *et al* (1966) gave a detailed description of these regions in the case of three spheres. The same regions now occur in the case of disks. Thus they will not be treated further. Only the result for I_{123} will be given:

$$
\begin{aligned}
\mathrm{I}_{123} &= \tfrac{1}{2}[\mathrm{I}_{12}+\mathrm{I}_{13}+\mathrm{I}_{23}-\pi+|T|/2], \\
T^2 &= [2(r_{12}^2 r_{13}^2 + r_{12}^2 r_{23}^2 + r_{13}^2 r_{23}^2) - (r_{12}^4 + r_{13}^4 + r_{23}^4)] \\
&= [4 r_{12}^2 r_{13}^2 - (r_{12}^2 + r_{13}^2 - r_{23}^2)^2].
\end{aligned}
\tag{2.2}
$$

I_{123} has been directly calculated by the author. It can also be determined by means of the results of Rowlinson (1963, 1964).

I_{123} is symmetrical in r_{12}, r_{13}, and r_{23} and can also be expressed in terms of the circumradius of the triangle (123), see Kratky (1976). A simpler expression can be obtained by introducing θ, the angle between $\boldsymbol{r}_{12}$ and $\boldsymbol{r}_{13}$. From the relation

$$r_{23}^2 = r_{12}^2 + r_{13}^2 - 2 r_{12} r_{13} \cos\theta \tag{2.3}$$

follows $T = 2 r_{12} r_{13} \sin\theta$, compare (2.2).

If the centres of the three disks lie on a straight line, the intersection of three disks can always be reduced to the intersection of two disks, cf Ree *et al* (1966):

$$\mathrm{I}(1,2,3) = \mathrm{I}(p,q), \qquad r_{pq} = \max(r_{12}, r_{13}, r_{23}). \tag{2.4}$$

Eventually, a lemma will be stated which deals with the fact that $\mathrm{I}(1,2,3)$ is a decreasing function of the distances in a certain sense.

Lemma 2.1. $\mathrm{I}^s(1,2,3) \subseteq \mathrm{I}^s(\overline{1,2,3})$ and thus $\mathrm{I}(1,2,3) \leqslant \mathrm{I}(\overline{1,2,3})$ if the triangle $(\overline{123})$ lies within the triangle (123). Only the relation $\mathrm{I}(1,2,3) \leqslant \mathrm{I}(\overline{1,2,3})$ is valid if $r_{ij} \geqslant \overline{r}_{ij}$, $1 \leqslant i < j \leqslant 3$.

3. The intersection of four disks

We consider the intersections $\mathrm{I}(1,2,3)$, $\mathrm{I}(1,2,4)$, $\mathrm{I}(1,3,4)$, and $\mathrm{I}(2,3,4)$ in order to investigate $\mathrm{I}(1,2,3,4)$. If one of those intersections $\mathrm{I}(i,j,k)$ is not I_{ijk}, then $\mathrm{I}(1,2,3,4)$ can be reduced straightforwardly to the intersection of three disks. For example, $\mathrm{I}(1,2,3) = \mathrm{I}(1,2)$ yields $\mathrm{I}(1,2,3,4) = \mathrm{I}(1,2,4)$, compare equations (1.2), (1.3).

Definition 3.1. Four disks are considered. The condition $\mathrm{I}(i,j,k) = \mathrm{I}_{ijk}$ (for any triple of the disks) will be called CI_4 in the following. The relation $\mathrm{I}(1,2,3,4) = \mathrm{I}_{1234}$ will be termed RI_4.

From the above considerations it follows that CI_4 is a necessary condition for RI_4. The case RI_4 is characterised by the fact that the convex area $\mathrm{I}^g(1,2,3,4)$ is bounded by four arcs which belong to the circles 1–4. In the special numbering of figure 1, one can easily deduce the following relations:

$$\mathrm{I}^g_{1234} = \mathrm{I}^g_{123} + \mathrm{I}^g_{124} - \mathrm{I}^g_{12}, \qquad \mathrm{I}^g_{1234} = \mathrm{I}^g_{134} + \mathrm{I}^g_{234} - \mathrm{I}^g_{34}. \tag{3.1}$$

It follows immediately that the same relations are also valid for I instead of I^g.

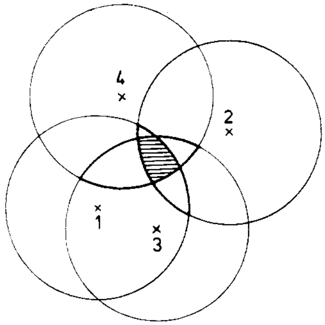

Figure 1. The area of intersection of four disks.

Theorem 3.1. CI_4 together with the condition that the four centres can be interpreted as the corners of a convex quadrilateral are necessary and sufficient for RI_4 to be valid.

Proof. CI_4 is a necessary condition of RI_4. If the second condition of the theorem is not fulfilled, one centre (e.g. centre 4) lies within the triangle formed by the others. It follows that $I(1,2,3,4)=I(1,2,3)$ since the supplementary restriction $r_{p4}<1$ (see definition 1.1) does not reduce $I^s(1,2,3)$ then. On the other hand, from the conditions of the theorem it follows that $I^s(1,2,3,4)\subset I^s(i,j,k)$ for any triple i, j, k of the four disks. Since it can also be shown that $I^s(1,2,3,4)\neq\varnothing$ in this case, it follows $I(1,2,3,4)=I_{1234}$ (compare definition 1.2).

The following lemma, which will be used in § 5, can be proved by means of arguments which are analogous to the preceding theorem.

Lemma 3.2. If centre 4 lies within the triangle (123), it follows that $I(1,2,3,4)=I(1,2,3)$. CI_4 is not required now. Even the more general conditions that r_{12}, r_{13} and r_{23} are at least 1 and $r_{i4}\leqslant 1$ $(1\leqslant i\leqslant 3)$ yield $I(1,2,3,4)=I(1,2,3)$.

In § 3, the problem of the determination of $I(1,2,3,4)$ has been solved completely. If CI_4 is not fulfilled, e.g. $I(1,2,4)=I(1,2)$, it follows that $I(1,2,3,4)=I(1,2,3)$. Examination of CI_4 and determination of $I(1,2,3)$ can be done according to § 2. If CI_4 is fulfilled and one centre (e.g. centre 4) lies within the triangle spanned by the others, $I(1,2,3,4)=I(1,2,3)=I_{123}$ follows in our example. Only the case RI_4 remains. The corners i and j shall be connected by a diagonal of the quadrilateral (1234). Then (3.1) can be generalised to give:

$$I^g_{1234}=I^g_{ikl}+I^g_{jkl}-I^g_{kl}, \qquad I_{1234}=I_{ikl}+I_{jkl}-I_{kl}. \tag{3.2}$$

Of course, k and l are also connected by a diagonal under the above assumption, yielding a second equation of type (3.2) (compare (3.1)).

$I(1,2,3,4)$ can be described as a function of r_{12}, r_{13}, r_{14}, θ_3 and θ_4, where θ_i is the angle between $\boldsymbol{r}_{12}$ and $\boldsymbol{r}_{1i}$. $I(1,2,3,4)$ remains unchanged if $\theta_i\to-\theta_i$ for $i=3,4$ conjointly. Thus, it is sufficient to consider θ_4 for example only in the region $0\leqslant\theta_4\leqslant\pi$ (modulo 2π). Table 1 shows the angular dependence of $I(1,2,3,4)$ in the case $r_{12}=1{\cdot}5$, $r_{13}=1{\cdot}0$, and $r_{14}=0{\cdot}5$.

Table 1. I(1, 2, 3, 4) as a function of θ_3 and θ_4 for $r_{12} = 1{\cdot}5$, $r_{13} = 1{\cdot}0$, and $r_{14} = 0{\cdot}5$. The disks have unit radius.

$\left(\frac{4}{\pi}\right)\theta_3$	$\left(\frac{4}{\pi}\right)\theta_4$	Type†	Value	$\left(\frac{4}{\pi}\right)\theta_3$	$\left(\frac{4}{\pi}\right)\theta_4$	Type	Value
0	0	I_{12}	0·45331				
	1	I_{124}	0·44459				
	2	I_{124}	0·28185	3–5	0–4	0	0·0
	3	I_{24}	0·05024				
	4	0	0·0				
1	0	I_{123}	0·36110	6	0	I_{123}	0·07756
	1	I_{123}	0·36110		1	I_{1234}	0·06885
	2	I_{1234}	0·27996		2	I_{234}	0·01331
	3	I_{24}	0·05024		3	I_{234}	0·00002
	4	0	0·0		4	0	0·0
2	0	I_{123}	0·07756	7	0	I_{123}	0·36110
	1	I_{123}	0·07756		1	I_{1234}	0·35239
	2	I_{123}	0·07756		2	I_{1234}	0·18964
	3	I_{234}	0·02446		3	I_{234}	0·03720
	4	0	0·0		4	0	0·0

† Compare definition 1.2.

4. The general case of n disks

In this section, the above considerations will be generalised to the case of n disks.

Definition 4.1. The condition $I(i, j, k) = I_{ijk}$ for any triple of n centres of disks ($n \geqslant 3$) is called CI_n. The relation $I(1, 2, \ldots, n) = I_{12\ldots n}$ is called RI_n.

CI_n is a necessary condition for RI_n; compare the case $n = 4$ in § 3. For $n = 3$, CI_n and RI_n are identical.

Definition 4.2. Assume n distinct points in a plane ($n \geqslant 3$), none of which lie on the straight line through any other two. Each pair of points is connected by a line segment. A convex polygon of m points enveloping the figure is generated. The polygon is called P_n^m ($3 \leqslant m \leqslant n$) with corners numbered $1, 2, \ldots, m$.

The m corners of P_n^m yield m pairs of neighbouring points of the polygon; the remaining $m(m-3)/2$ pairs are connected by a diagonal of the polygon.

Lemma 4.1. On the presuppositions of definition 4.2, $P_n^m = P_n^n$ if and only if no point lies within the triangle formed by any triple out of the $n-1$ other points.

In § 3, $n = 4$ has been studied. In the case CI_4, two possibilities have been found; either the four centres form a convex quadrilateral, or one point lies within the triangle spanned by the others. Now, these cases can be classified as P_4^4 and P_4^3, respectively. There is no other possibility for $n = 4$ since $3 \leqslant m \leqslant n$ (compare definition 4.2) follows from the condition that the n points are distinct and none of the n points lie on the straight line through two other points, for which CI_4 is a sufficient condition, cf equation (2.4).

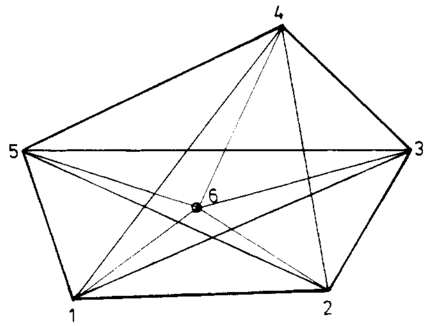

Figure 2. An example for P_6^5 (see definition 4.2).

Lemma 4.2. If the centres of n disks form a polygon P_n^m with $m<n$, it follows that $I(1, 2, \ldots, n) = I(1, 2, \ldots, m)$.

Thus, it is sufficient to concentrate on the case P_n^n in order to study RI_n, i.e. $I_{12\ldots n}$.

Theorem 4.3. CI_n together with the condition that the n centres form P_n^n are necessary and sufficient for RI_n.

Proof. The first part of the proof ('necessary') has already been outlined. This will now be done for the second part ('sufficient'). In analogy to theorem 3.1, it can be shown that $I^s(1, 2, \ldots, n) \subset I^s(\beta)$, $\{\beta\}$ being a set of $n-1$ centres arbitrarily selected from the n centres. Furthermore, $I^s(1, 2, \ldots, n) \neq \varnothing$. This results in $I(1, 2, \ldots, n) = I_{12\ldots n}$ (see definition 1.2).

Theorem 4.4. The intersection of n disks can always be reduced to contributions from intersections of less than four disks. It can be evaluated explicitly.

Proof (by complete induction). For $n \leqslant 3$, nothing has to be proved. Now, it will be assumed that the proposition is true for all numbers of centres less than n $(n>3)$. It will be shown that the proposition is then also true for n disks. If CI_n is not fulfilled or $m<n$ is valid for P_n^m, a direct reduction of $I(1, 2, \ldots, n)$ to an intersection of $\bar{n}$ disks is possible $(\bar{n}<n)$, see lemma 4.2. The proposition follows from this. Therefore, only the case RI_n has to be considered further. In this case, two corners (e.g. numbered 1, 2) are selected from P_n^n which are connected by a diagonal of P_n^n. Then, there exist at least two other points (e.g. called 3, 4) of P_n^n which also define a diagonal of P_n^n and which fulfill the following condition: the line segment connecting 1 and 2 is also a diagonal of the (convex) quadrilateral which is formed by the points 1–4. It follows that $I^g(1, 2, 3, 4) = I^g_{1234} = I^g_{134} + I^g_{234} - I^g_{34}$ (compare (3.1), (3.2)). Restricting both sides of the equation additionally to $I^g_{5\ldots n}$ yields

$$I^g_{1234\ldots n} = I^g_{134\ldots n} + I^g_{234\ldots n} - I^g_{34\ldots n}. \tag{4.1}$$

In the consequence, the same relation holds for $I_{1234\ldots n}$. Since the intersection of less than n disks can be reduced further due to the assumption, $I_{1234\ldots n}$ can be represented finally as a sum (difference) of intersections of less than four disks. These intersections are known explicitly.

If a diagonal connects the centres i and j, (4.1) can be generalised to

$$\mathrm{I}^g_{12\ldots hijk\ldots n} = \mathrm{I}^g_{12\ldots hik\ldots n} + \mathrm{I}^g_{12\ldots hjk\ldots n} - \mathrm{I}^g_{12\ldots hk\ldots n}. \tag{4.2}$$

P^n_n has $n(n-3)/2$ diagonals. Thus one has $n(n-3)/2$ equations of type (4.2) which yield the same $\mathrm{I}_{1234\ldots n}$. Eventually, from this it follows that the representation of $\mathrm{I}_{1234\ldots n}$ as a sum (difference) of intersections of less than four disks is not unique. This results in a set of relations between these intersections. In the case of $n=4$, there is one relation, i.e.

$$\mathrm{I}_{123} + \mathrm{I}_{124} - \mathrm{I}_{12} = \mathrm{I}_{134} + \mathrm{I}_{234} - \mathrm{I}_{34} \tag{4.3}$$

if 1 and 2 are connected by a diagonal of P^4_4, see (3.1).

5. Physical applications

In the case of hard disks, the area of intersection of disks occurs as an integrand in the Mayer cluster integrals, see § 1. To be more accurate, the so called complete star graph ϕ_n contributing to the virial coefficient B_n has the integrand $\mathrm{I}(1, 2, \ldots, n-1)$. The other clusters are simpler, containing products of $\mathrm{I}(1, 2, \ldots, m)$ with $m < n-1$ (Kratky 1976). If these intersections were not known, all Mayer clusters contributing to B_n would consist in $(2n-3)$-fold integration in the case of disks. Using the results of this paper, $2n-5$ variables of integration occur for ϕ_n, at most $2n-7$ variables for the other integrals. Thus, a great portion of the cluster integrals is simplified considerably and can be evaluated numerically for $n \leq 7$. Incidentally, the Mayer cluster expansion up to B_7 was exhibited by Hoover and DeRocco (1962).

B_3 and B_4 are even known analytically for hard disks (Rowlinson 1964, Hemmer 1965). B_5 is known accurately (Kratky 1976). When determining B_5, the author used $\mathrm{I}(1, 2, 3)$. This was sufficient for all clusters except ϕ_5, yielding at most three-fold integration which was carried out numerically. Since ϕ_5 is a definite function of the other clusters in the case of hard disks (Ree and Hoover 1964, Kratky 1976), it was not necessary to calculate ϕ_5 directly when evaluating B_5. With the help of the present paper, it will also be possible to increase the accuracy of B_6. All Mayer clusters except ϕ_6 are now accessible to numerical integration with reasonable accuracy. Again, ϕ_6 is a definite function of the other clusters. Thus, one can avoid the difficulty that ϕ_6 consists of a seven-fold integration, the integrand being $\mathrm{I}(1, 2, 3, 4, 5)$.

The Mayer cluster integrals can be combined to a new set of integrals, the so called Hoover graphs (Ree and Hoover 1964, 1967, Kilpatrick 1971, Kratky 1976). One of the five Hoover graphs contributing to B_5 has the value zero for hard disks. This graph will be denoted (III) in the following according to Kratky (1976). That the value of (III) is zero has been shown by Ree and Hoover (1964). It is also a consequence of lemma 3.2. The integrand of (III) can be interpreted as $\mathrm{I}(1, 2, 3, 4) - \mathrm{I}(1, 2, 3)$. It is restricted to a region where lemma 3.2 can be used. Since (III) is a linear combination of Mayer clusters including ϕ_5, from (III) $= 0$ it follows that ϕ_5 is a function of the other Mayer cluster integrals (see above).

A further application of the results of this paper lies in the field of distribution functions which are also connected with the equation of state (Rowlinson 1963, Hemmer 1965, Ree *et al* 1966). In the theory of Born, Green, and Yvon (BGY), the superposition approximation yields deviations from the correct triplet distribution function, $-\mathrm{I}(1, 2, 3)$ being the first correction term (Rowlinson 1963, Lee *et al* 1968, 1969). This is valid for instance for disks and spheres. BGY2, an improvement of BGY theory, has been proposed by Lee *et al* (1968, 1971). It has been applied to hard spheres with remarkable success. The better knowledge of $\mathrm{I}(1, 2, \ldots n)$ in the case of disks due to the present work can help when considering BGY2 for hard disks.

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